

Review of Calculus Concepts

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AP Calculus

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Precalculus

$$\text{Slope: } m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Point-Slope equation: } y - y_1 = m(x - x_1)$$

$$\text{Slope intercept equation: } y = mx + b$$

$$\text{Distance Formula: } d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Even & Odd Functions:

A function is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.

Intervals:

$a < x < b$	Bounded open interval	(a, b)
$a < x$	Unbounded open interval	(a, ∞)
$x < b$	Unbounded open interval	$(-\infty, b)$
$a \leq x \leq b$	Bounded closed interval	$[a, b]$
$a \leq x$	Unbounded closed interval	$[a, \infty]$
$x \leq b$	Unbounded closed interval	$(-\infty, b]$

$$\text{Composition: } (f \circ g)(x) = f(g(x))$$

Shift Formulas ($c > 0$)

$$\begin{array}{ll} \text{Vertical Shifts of } y = f(x): & y = f(x) + C \quad (\text{Shifts } f(x) \text{ up } C \text{ units}) \\ & y = f(x) - C \quad (\text{Shifts } f(x) \text{ down } C \text{ units}) \end{array}$$

$$\begin{array}{ll} \text{Horizontal Shifts of } y = f(x): & y = f(x - c) \quad (\text{Shifts } f(x) \text{ right } C \text{ units}) \\ & y = f(x + c) \quad (\text{Shifts } f(x) \text{ left } C \text{ units}) \end{array}$$

Stretch & Shrink Formulas ($c > 0$)

Vertical Stretches and Shrinks of $y = f(x)$:

$$\begin{array}{ll} y = cf(x), c > 1 & \text{Stretches the graph vertically by a factor of } C \\ y = cf(x), 0 < c < 1 & \text{Shrinks the graph vertically by a factor of } C \end{array}$$

Horizontal Shrinks and Stretches of $y = f(x)$:

$$\begin{array}{ll} y = f(cx), c > 1 & \text{Shrinks the graph horizontally by a factor of } 1/c \\ y = f(cx), 0 < c < 1 & \text{Stretches the graph horizontally by a factor of } 1/c \end{array}$$

Limits

The limit of $f(x)$ as x approaches c equals L .

$$\lim_{x \rightarrow c} f(x) = L$$

One-sided Limits: If $\lim_{x \rightarrow c^-} f(x) = L$ and $\lim_{x \rightarrow c^+} f(x) = L$ then $\lim_{x \rightarrow c} f(x) = L$ exists.

Limit Theorems:

Limit of a Constant Function: If $f(x) = c$, then $\lim_{x \rightarrow a} f(x) = c$

Addition: $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = F + G$

Subtraction: $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = F - G$

Multiplication: $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} g(x) \right] = F \cdot G$

Division: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{F}{G} \quad (G \neq 0)$

Product of a Constant and a Limit: $\lim_{x \rightarrow a} [c \cdot g(x)] = c \lim_{x \rightarrow a} g(x) = cG$

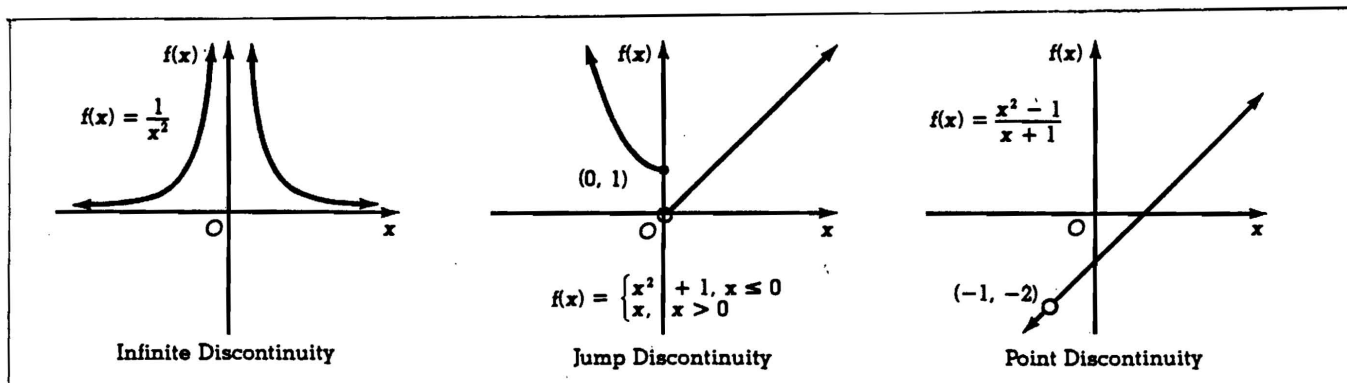
Powers: $\lim_{x \rightarrow a} [f(x)^n] = \left[\lim_{x \rightarrow a} f(x) \right]^n = F^n$

Roots: $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{F}$

Limit of the Composite: $\lim_{x \rightarrow a} f[g(x)] = f\left[\lim_{x \rightarrow a} g(x)\right] = f(G)$

Continuity: A function f is continuous on an interval if it is continuous at each point of the interval.

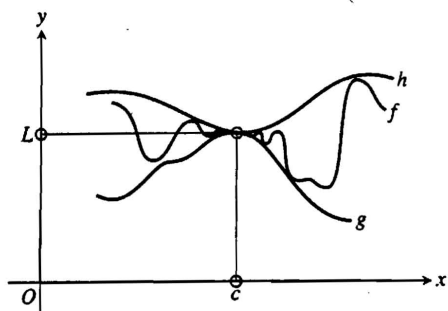
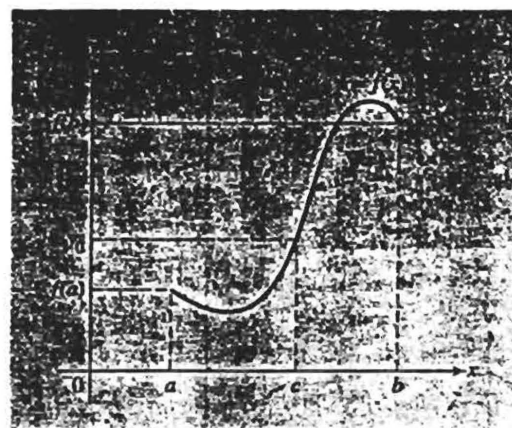
Discontinuities:



The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

If y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \neq c$ in some interval about c and that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L. \text{ Then } \lim_{x \rightarrow c} f(x) = L.$$

Special Cases

$$\lim_{x \rightarrow 0} a^x = 1 \quad a > 0$$

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = e$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \rightarrow \infty} \sqrt[x]{x} = 1$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{y}{x}\right)^x = e^y$$

$$\lim_{x \rightarrow \infty} \frac{a^x - 1}{x} = \ln a \quad a > 0$$

$$\lim_{m \rightarrow \infty} \frac{a^m}{m!} = 0$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \rightarrow \infty} \frac{x^m}{e^x} = 0$$

$$\lim_{x \rightarrow \infty} \frac{(\ln x)^m}{x} = 0$$

$$\lim_{x \rightarrow 1} \frac{x-1}{\ln x} = 1$$

$$\lim_{x \rightarrow \infty} \frac{\ln(x+1)}{x} = 0$$

$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$	$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$	$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$
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Derivatives

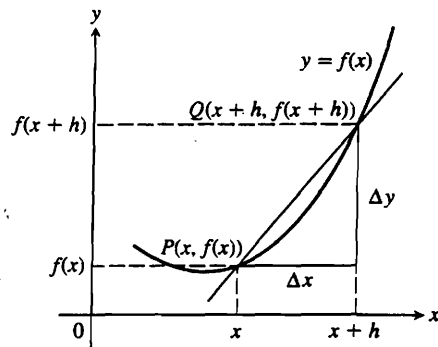
Derivatives deal with the rate of change in a function. Geometrically, the derivative gives the slope of the tangent line to the curve.

(a) First derivative of $y = f(x)$ with respect to x is defined as



Alternative notations are $f'(x)$, $\frac{df(x)}{dx}$, and y' . If $y = f(t)$,

$$\frac{dy}{dt} = \frac{df(t)}{dt} = j$$



(b) Second and higher derivatives of the same function are

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[f'(x)] = y''$$

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d}{dx}[f^{(n-1)}(x)] = f^{(n)}(x)$$

RATES OF CHANGE

The average velocity of a body moving along a line from position $s = f(t)$ to position $s + \Delta s = f(t + \Delta t)$ is $v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$.

Instantaneous velocity is the derivative of position, or distance. If the position function of a body moving along a line is $s = f(t)$, the body's (instantaneous) velocity at time t is $v(t) = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$.

Speed is the absolute value of velocity.

Acceleration is the derivative of velocity. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

The average rate of change of a function $f(x)$ over the interval from x to $x+h$ is:

$$\text{Average rate of Change} = \frac{f(x+h) - f(x)}{h}$$

The (instantaneous) rate of change of f at x is the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Implicit Differentiation:

1. Differentiate both sides of the equation with respect to x .
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx .
4. Solve for dy/dx by dividing.

Applications of Derivatives

(1) Rolle's Theorem

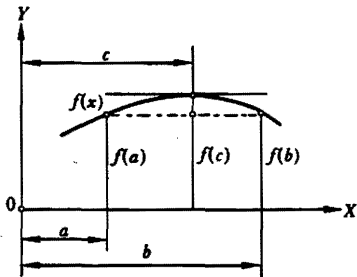
If a function $f(x)$ is continuous in the closed interval $[a, b]$ and is differentiable in the open interval (a, b) and if $f(a) = f(b)$, then there is at least one point $(x = c)$ in (a, b) in which

$$f'(c) = 0$$

(2) Lagrange's Theorem (first mean-value theorem)

If a function $f(x)$ is continuous in the closed interval $[a, b]$ and is differentiable in the open interval (a, b) , then there is at least one point $(x = c)$ in (a, b) in which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



The Procedure for Newton's Method

1. Guess a first approximation of a root for the equation $f(x) = 0$. A graph of $y = f(x)$ will help.
2. Use the first approximation to get a second, the second to get a third, and so on. To go from the n th approximation x_n to the next approximation x_{n+1} , use the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1)$$

where $f'(x_n)$ is the derivative of f at x_n .

L'Hôpital's Rules

$f(x)$ and $g(x)$ are two continuous functions of x having continuous derivatives at x .

If $f(x)/g(x)$ for $x \rightarrow a$ is $0/0$ or ∞/∞ ,

$$\text{then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

If $f(x)g(x)$ for $x \rightarrow a$ is $(0)(\infty)$ or $(\infty)(0)$, then

$$\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} \frac{f'(x)}{[1/g(x)]'}$$

If $f(x) - g(x)$ for $x \rightarrow a$ is $\infty - \infty$,

then

$$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} \frac{[1/f(x) - 1/g(x)]'}{[1/f(x) - 1/g(x)]'}$$

If $f(x)^{g(x)}$ for $x \rightarrow a$ is 0^0 or ∞^0 or 1^∞ , then

$$\lim_{x \rightarrow a} f(x)^{g(x)} = \lim_{x \rightarrow a} e^{g(x) \ln f(x)}$$

Summary of Curve-Sketching Techniques

1. Compute $f'(x)$ and $f''(x)$.
2. Find all maximum and minimum points.
 - (a) Set $f'(x) = 0$ and solve for x . Suppose that $f'(a) = 0$.
 - (i) If $f''(a) > 0$, the curve is concave up at $x = a$, and so $f(x)$ has a minimum at $x = a$. The minimum point is $(a, f(a))$.
 - (ii) If $f''(a) < 0$, the curve is concave down at $x = a$, and so $f(x)$ has a maximum at $x = a$. The maximum point is $(a, f(a))$.
 - (iii) If $f''(a) = 0$, examine $f'(x)$ to the left and right of $x = a$ in order to determine if the function changes from increasing to decreasing or vice versa.
 - (b) Make a partial sketch of the graph near each point where $f(x)$ has a horizontal tangent line.
3. Determine the concavity of $f(x)$.
 - (a) Set $f''(x) = 0$ and solve for x . Suppose that $f''(b) = 0$. Next, test the concavity for x near b . If the concavity changes at $x = b$, then $(b, f(b))$ is an inflection point. Otherwise the concavity at $x = b$ is the same as at other nearby points.
4. Consider other properties of the function and complete the sketch.
 - (a) If $f(x)$ is defined at $x = 0$, the y -intercept is $(0, f(0))$.
 - (b) Does the partial sketch suggest that there are x -intercepts? If so, they are found by setting $f(x) = 0$ and solving for x . (Solve only in easy cases or when a problem essentially requires you to calculate the x -intercepts.)
 - (c) Observe where $f(x)$ is defined. Sometimes the function is given only for restricted values of x . Sometimes the formula for $f(x)$ is meaningless for certain values of x .
 - (d) Look for possible asymptotes.
 - (i) Examine the formula for $f(x)$. If some terms become insignificant as x gets large and if the rest of the formula gives the equation of a straight line, then that straight line is an asymptote.
 - (ii) Suppose that there is some point a such that $f(x)$ is defined for x near a but not at a (for example, $1/x$ at $x = 0$). If $f(x)$ gets arbitrarily large (in the positive or negative sense) as x approaches a , then the vertical line $x = a$ is an asymptote for the graph.
 - (e) Complete the sketch.

<p>$y' > 0 \Rightarrow$ graph rises from left to right; may be wavy</p>	<p>$y' < 0 \Rightarrow$ graph falls from left to right; may be wavy</p>
<p>$y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall</p>	<p>$y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall</p>
<p>$y' = 0$ and $y'' < 0$ at a point; graph has local minimum</p>	<p>$y' = 0$ and $y'' > 0$ at a point; graph has local maximum</p>
<p>Inflection point</p>	<p>y' changes sign \Rightarrow graph has local maximum or minimum</p>

Differentiation

1. Derivative of a constant : $\frac{d}{dx} (c) = 0$
2. Constant Multiple Rule : $\frac{d}{dx} [k \cdot f(x)] = k \cdot \frac{d}{dx} [f(x)]$
3. Sum Rule : $\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} [f(x)] + \frac{d}{dx} [g(x)]$
4. General Power Rule : $\frac{d}{dx} \{ [g(x)]^r \} = r \cdot [g(x)]^{r-1} \cdot \frac{d}{dx} [g(x)]$
5. Product Rule : $\frac{d}{dx} [f(x) g(x)] = f(x) g'(x) + g(x) f'(x)$
6. Quotient Rule : $\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \left(\frac{f}{g} \right)'(x) = \frac{g(x) f'(x) - f(x) g'(x)}{[g(x)]^2}$
7. Chain Rule : $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ or $\frac{d}{dx} [f(u)] = \frac{d}{du} [f(u)] \frac{du}{dx}$
8. Chain Rule (Prime Notation) : $\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$
9. Fractional Exponents : $\frac{d}{dx} [x^{1/n}] = \frac{1}{n} x^{(1/n)-1}$ and $\frac{d}{dx} [x^{p/q}] = \frac{p}{q} x^{(p/q)-1}$
10. The Reciprocal Rule : $\frac{d}{dx} \left[\frac{1}{g(x)} \right] = \left(\frac{1}{g} \right)'(x) = - \frac{g'(x)}{[g(x)]^2}$
11. Differentiating the Trigonometric Functions :

$$\frac{d}{dx} (\sin x) = \cos x$$

$$\frac{d}{dx} (\sin u) = \cos u \frac{du}{dx}$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

$$\frac{d}{dx} (\cos u) = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

$$\frac{d}{dx} (\tan u) = \sec^2 u \frac{du}{dx}$$

$$\frac{d}{dx} (\cot x) = -\csc^2 x$$

$$\frac{d}{dx} (\cot u) = -\csc^2 u \frac{du}{dx}$$

$$\frac{d}{dx} (\sec x) = \sec x \tan x$$

$$\frac{d}{dx} (\sec u) = \sec u \tan u \frac{du}{dx}$$

$$\frac{d}{dx} (\csc x) = -\csc x \cot x$$

$$\frac{d}{dx} (\csc u) = -\csc u \cot u \frac{du}{dx}$$

Robbins (1998)

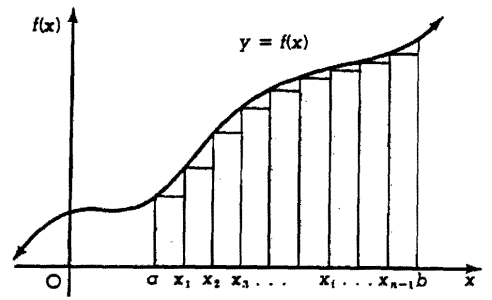
Integrals

The integral is connected with the idea of area; this is expressed by the definite integral of a function.

Definite Integral

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} f(x_i) \Delta x = \left[\int f(x) dx \right]_a^b = [F(x)]_a^b = F(b) - F(a)$$

The numbers a and b are called the lower and upper limits of integration, and $[a, b]$ is called the range of integration. Geometrically, the definite integral of $f(x)$ with respect to x , between limits $x=a$ to $x=b$ is the area bounded by $f(x)$, the x -axis, and the verticals through the end points of a and b .



$$A_n = \sum_{i=1}^n f(x_i)(x_{i-1} - x_i)$$

Rules of Limits

$$\int_a^b = -\int_b^a$$

$$\int_a^b + \int_b^c = \int_a^c$$

$$\int_a^c - \int_b^c = \int_a^b$$

$$\int_a^a = 0$$

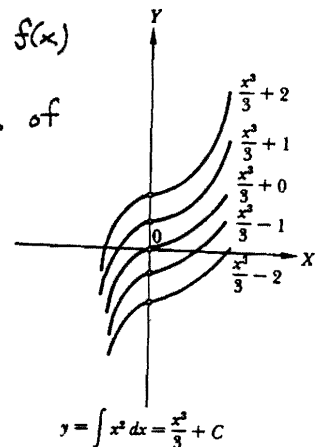
The Indefinite Integral

$F(x)$ is an indefinite integral (antiderivative) of $f(x)$ if $\frac{dF(x)}{dx} = f(x)$

Since the derivative of $F(x) + C$ is also equal to $f(x)$, all integrals of $f(x)$ are included in the expression

$$\int f(x) dx = F(x) + C$$

f is called the integrand and C is an arbitrary constant. Because of the indeterminacy of C , there is an infinite number of $F(x) + C$, differing by their relative position to the x -axis.



The Mean Value Theorem for Definite Integrals

If f is continuous on the closed interval $[a, b]$, then, at some point c in the interval $[a, b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

The Fundamental Theorem of Calculus (1)

If f is continuous on $[a, b]$, then the function $F(x) = \int_a^x f(t) dt$ has a derivative at every point in $[a, b]$, and $\frac{dF}{dx} = \frac{d}{dx} \int_a^x f(t) dt = f(x)$

The Fundamental Theorem of Calculus (2)

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Numerical Integration: The Trapezoidal Rule and Simpson's Method

THE TRAPEZOIDAL RULE

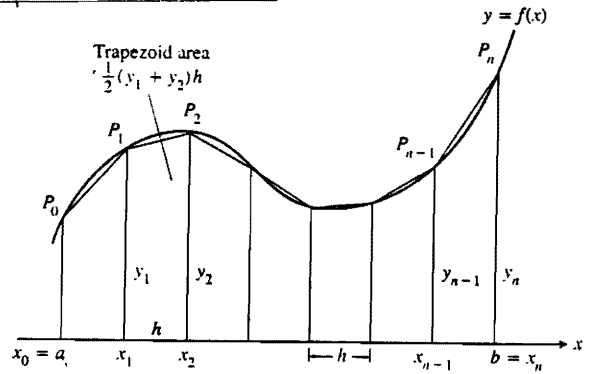
To approximate

$$\int_a^b f(x) dx,$$

use

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n) \quad (1)$$

with $[a, b]$ partitioned into n subintervals of length $h = (b - a)/n$.



The Trapezoidal Rule approximates short stretches of curve with line segments. To estimate the area under the curve, we add the areas of the trapezoids formed by joining the ends of these segments to the x -axis.

SIMPSON'S RULE

To approximate

$$\int_a^b f(x) dx,$$

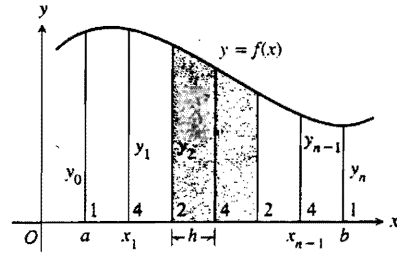
use

$$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 2y_{n-2} + 4y_{n-1} + y_n) \quad (5)$$

with $[a, b]$ partitioned into an *even number*, n , of subintervals of length $h = (b - a)/n$.

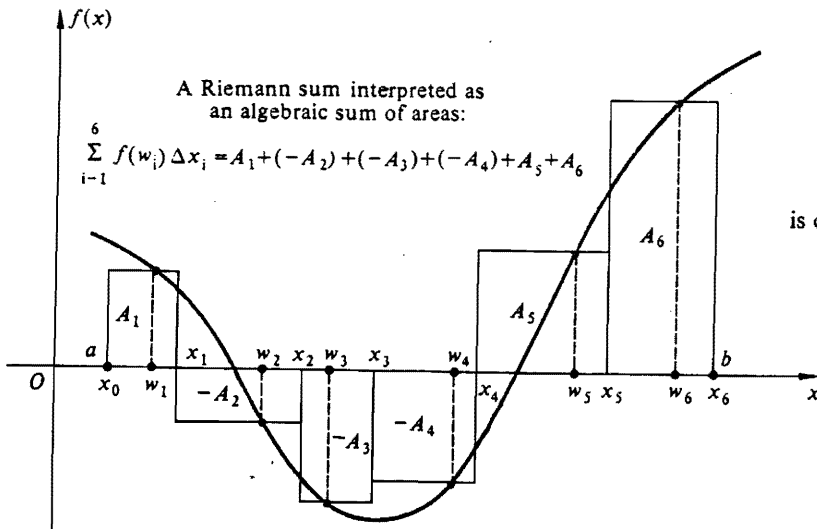
The y 's in Eq. (5) are the values of $y = f(x)$ at the partition points

$x_0 = a, x_1 = a + h, x_2 = a + 2h, \dots, x_{n-1} = a + (n - 1)h, b = x_n$



The y 's in Eq. (5) are the values of f at the points of subdivision.

The Integral as the Limit of Riemann Sums



$$\sum_{i=1}^n f(w_i) \Delta x_i = f(w_1) \Delta x_1 + f(w_2) \Delta x_2 + \dots + f(w_n) \Delta x_n$$

is called a *Riemann sum* for f on $[a, b]$.

U-Substitution in Definite Integrals

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

$$, u = g(x); du = g'(x) dx$$

Integration

1. Zero Rule : $\int_c^c f(x) dx = 0$
2. Integrating the other direction : $\int_b^a f(x) dx = -\int_a^b f(x) dx$
3. Constant Function Rule : $\int_a^b k dx = k(b-a)$
4. Constant Multiple Rule : $\int_a^b k f(x) dx = k \int_a^b f(x) dx$
5. Sum Rule : $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$
6. Difference Rule : $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$
7. Exponential Rule : $\int_a^b x^n dx = \left[\frac{1}{n+1} x^{n+1} \right]_a^b = \frac{1}{n+1} (b^{n+1} - a^{n+1})$
8. Domination Rule : $g(x) \geq f(x)$ on $[a, b]$ then $\int_a^b g(x) dx \geq \int_a^b f(x) dx$
- Interval Addition Rule : $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
10. Interval Subtraction Rule : $\int_a^c f(x) dx - \int_a^b f(x) dx = \int_b^c f(x) dx$
11. Differentiating the Integral : $\frac{d}{dx} \left(\int_a^u f(t) dt \right) = f(u) \frac{du}{dx}$
 $\frac{d}{dx} \left(\int_u^v f(t) dt \right) = f(v) \frac{dv}{dx} - f(u) \frac{du}{dx}$

12. Integrating the Trigonometric Functions :

$$\int \cos x dx = \sin x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

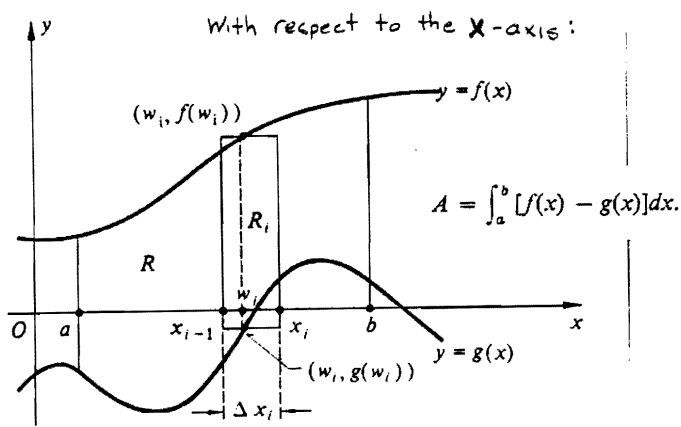
$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

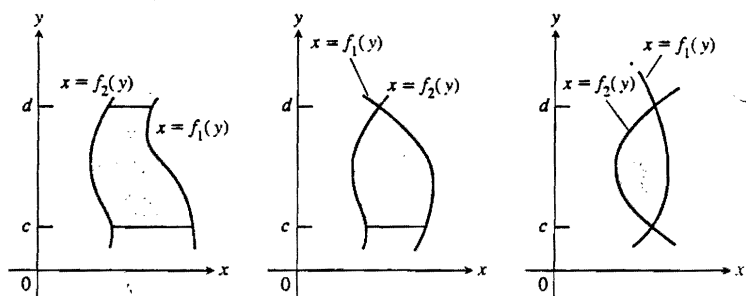
$$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

Region of Area bounded by Curves



For regions like these,

With respect to the y -axis:



use the formula

$$\text{Area} = \int_c^d (f_1(y) - f_2(y)) dy.$$

Volume By Parallel Cross sections : Discs and Washers

THE DISK METHOD

Volume of a Solid of Revolution (about the x -axis): $V = \int_a^b \pi [f(x)]^2 dx$

Volume of a Solid of Revolution (about the y -axis): $V = \int_c^d \pi [f(y)]^2 dy$

THE WASHER METHOD

Washer method about x -axis: $V = \int_a^b \pi ([f(x)]^2 - [g(x)]^2) dx$

Washer method about y -axis: $V = \int_c^d \pi ([F(y)]^2 - [G(y)]^2) dy$

Summary: $V = \int_{-}^{+} \pi [(outer\ radius)^2 - (inner\ radius)^2] d_{-}$

THE SHELL METHOD

Shell method about x -axis: $V = \int_c^d 2\pi y [F(y) - G(y)] dy$

shell method about y -axis: $V = \int_a^b 2\pi x [f(x) - g(x)] dx$

Summary: $V = \int_{-}^{+} 2\pi (\text{radius of cylinder}) (\text{height of cylinder}) d_{-}$

PARALLEL CROSS SECTIONS

If the cross-sectional area, $A(x)$, varies continuously with x , then we can find the volume of the solid by integrating $A(x)$ from a to b :

$$V = \int_a^b A(x) dx = (\text{average cross-sectional area}) \cdot (b-a)$$

The Transcendental Functions

THE LOGARITHM FUNCTION

The Natural Logarithm Function:

$$\ln x = \int_1^x \frac{1}{t} dt \quad (x > 0)$$

Basic properties for $\ln x$:

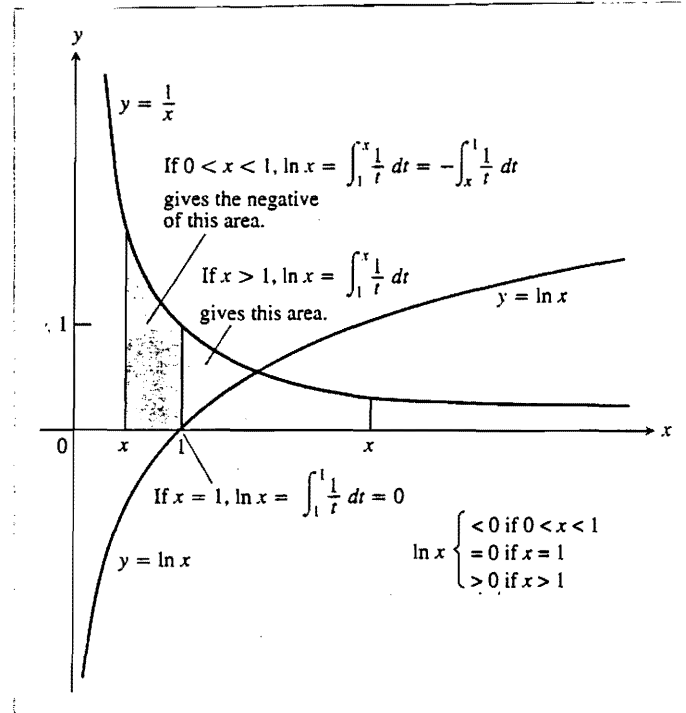
1. $\ln ax = \ln a + \ln x$
2. $\ln \frac{a}{x} = \ln a - \ln x$
3. $\ln \frac{1}{x} = -\ln x$
4. $\ln x^n = n \ln x$

$$\ln e = \int_1^e \frac{dt}{t} = 1$$

$$\ln 1 = 0$$

Change of Base:

$$\log_a x = \frac{\ln x}{\ln a}$$



$$\frac{d}{dx} (\ln|x|) = \frac{1}{x} \quad (x \neq 0)$$

$$\frac{d}{dx} [\ln g(x)] = \frac{1}{g(x)} \cdot \frac{d}{dx} g(x) = \frac{g'(x)}{g(x)}$$

$$\int \frac{du}{u} = \ln|u| + C$$

Integrating the Trigonometric Functions using the Natural Logarithm Function:

$$\int \tan u \, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C = -\ln|\csc u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + C = \ln|\csc u - \cot u| + C$$

Derivatives of Inverse Trigonometric Functions:

$$\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \cos^{-1} u = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \cot^{-1} u = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \csc^{-1} u = -\frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Integrals Leading to Inverse Trig. Functions:

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} |u| + C$$

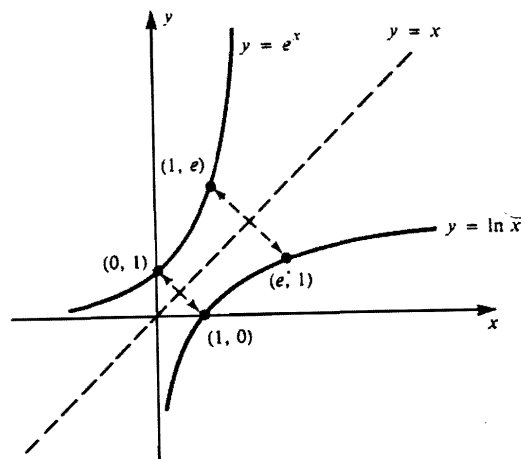
Robbins (1998)

THE EXPONENTIAL FUNCTION

$$\log_e x = \ln x \quad \text{If } y = \ln x, \text{ then } e^y = x.$$

The Natural Exponential Function:

$$f(x) = e^x, \text{ where } e = 2.7183\dots$$



Properties of Exponents:

$$1. e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$$

$$2. \frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$$

$$3. e^{-x} = \frac{1}{e^x}$$

$$\ln e^x = x \quad e^{\ln x} = x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^u) = \frac{d}{du}(e^u) \frac{du}{dx} = e^u \frac{du}{dx}$$

$$\int e^u du = e^u + C$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

The Function $y = a^x$

If a is a positive number, then $a^x = e^{x \ln a}$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

Exponential Rule

$$\text{If } a > 0, \text{ then } \frac{d}{dx} a^x = a^x \ln a; \quad \frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$$

$$\log_a a^x = x$$

$$a^{\log_a x} = x$$

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a} \frac{d}{dx} \ln x = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}$$

The Law of Exponential Change: $y = y_0 e^{kt}$

Radioactive Decay Equation: $y = y_0 e^{-kt}$

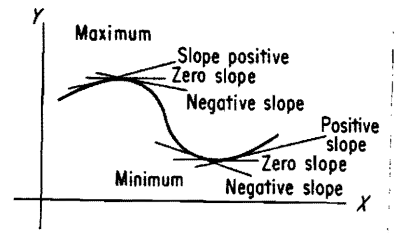
$$\text{Half Life} = \frac{\ln 2}{k}$$

For each positive integer n , $\left(1 + \frac{1}{n}\right)^n \leq e \leq \left(1 + \frac{1}{n}\right)^{n+1}$.

First Derivative Rule:

If $f'(a) > 0$, then $f(x)$ is increasing at $x=a$. If $f'(a) < 0$, then $f(x)$ is decreasing at $x=a$.

Second Derivative Rule: If $f''(a) > 0$, then $f(x)$ is concave up at $x=a$. If $f''(a) < 0$, the second derivative rule gives no info.



Conditions on the derivatives	Description of $f(x)$ at $x=a$	Graph of $y=f(x)$ near $x=a$
1. $f'(a)$ positive $f''(a)$ positive	$f(x)$ increasing $f(x)$ concave up	
2. $f'(a)$ positive $f''(a)$ negative	$f(x)$ increasing $f(x)$ concave down	
3. $f'(a)$ negative $f''(a)$ positive	$f(x)$ decreasing $f(x)$ concave up	
4. $f'(a)$ negative $f''(a)$ negative	$f(x)$ decreasing $f(x)$ concave down	

Locating Extreme Points ; Critical Values

The tangent line at a maximum or minimum point of a function $f(x)$ has zero slope.

• Look for possible extreme points of $f(x)$ by setting $f'(x) = 0$ and solving for x .

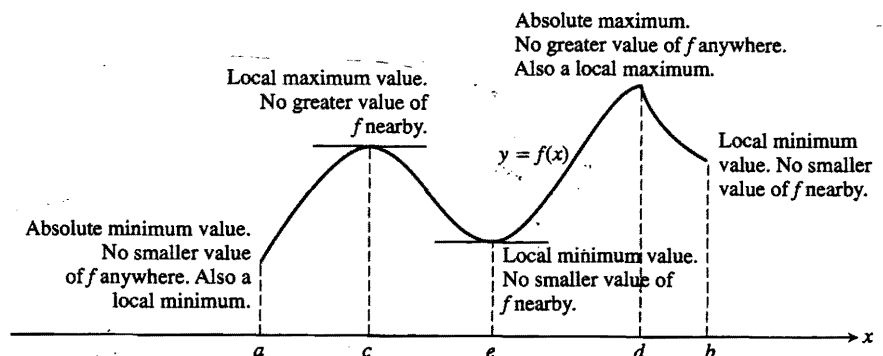
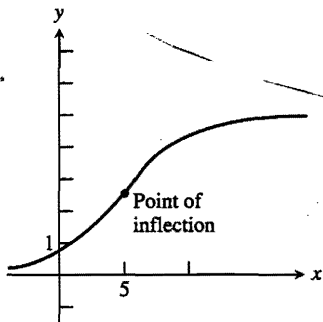
Plug in that value(s) of x into the original function, $f(x)$, to obtain the exact coordinates of the extreme point(s).

An extreme point may be a maximum or minimum value of the function. The second derivative determines concavity.

Locating Inflection Points

An inflection point of a function $f(x)$ can occur at a value of x for which $f''(x)$ is zero, because the curve is concave up where $f''(x)$ is positive and concave down where $f''(x)$ is negative. An inflection point is the point where concavity changes.

• Look for possible inflection points by setting $f''(x) = 0$ and solving for x .



The Second Derivative Test for Concavity

The graph of a twice-differentiable function $y = f(x)$ is
 concave down on any interval where $y'' < 0$,
 concave up on any interval where $y'' > 0$.

Curve Sketching

- Look for possible extreme points of $f(x)$ by setting $f'(x) = 0$ and solving for x . Plug in that value of x into $f(x)$ to obtain exact coordinates.
- Plug in that same value of x into $f''(x)$ to obtain the concavity.
- Look for possible inflection points by setting $f''(x) = 0$ and solving for x .

Velocity & Acceleration

$x(t)$ = position function
 $v(t)$ = velocity function
 $a(t)$ = acceleration function

$$a(t) = v'(t) = x''(t)$$

Galileo's Formula: $x(t) = -\frac{1}{2}gt^2 + v_0t + x_0$

v_0 = initial velocity
 x_0 = initial height
 $g = 32\text{ft/s}^2$

• A particle is at rest when $v(t) = 0$

$$\text{Average value of } f = \frac{1}{b-a} \int_a^b f(x) dx \qquad \text{Average rate of change} = \frac{f(b) - f(a)}{b-a}$$

Integration by Parts

• An effective and versatile method of integration that can be applied to certain integrals is known as integration by parts.

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

or, if $u = f(x)$ and $v = g(x)$,

The Integration-by-Parts Formula

$$\int u dv = uv - \int v du$$

• The formula for integration by parts is obtained easily from the product rule for differentiation. $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$. Integrating both sides yields $uv = \int v\frac{du}{dx} dx + \int u\frac{dv}{dx} dx = \int v du + \int u dv$.